BRIEF COMMUNICATIONS

SOLUTION OF SOME NONLINEAR HEAT TRANSFER PROBLEMS

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We will consider some heat transfer problems governed by a nonlinear relation introduced by radiation in accordance with the Stefan-Boltzmann law.

The problems are solved by the method of finite differences which leads to specific schemes depending on the nonlinear quantities in the equation or the boundary conditions. Suppose we have the nonlinear equation

$$\frac{\partial u}{\partial t}=A(x,\ t)\frac{\partial^2 u}{\partial x^2}+B(x,\ t)\frac{\partial u}{\partial x}+f(x,\ t,\ u),$$

the linear boundary conditions

$$\alpha_1\frac{\partial u}{\partial x}+\beta_1 u=\gamma_1(t)\quad \text{for}\ x=a,$$

$$\alpha_2\frac{\partial u}{\partial x}+\beta_2 u=\gamma_2(t)\quad \text{for}\ x=b$$

and the initial condition

$$u(x,\ 0)=\varphi(x).$$

We assume that the conditions $A > 0$ are satisfied and the functions A, B, f, A_x, A_{xx} are bounded and uniformly continuous with respect to all the arguments. Moreover, the Lipschitz condition

$$|f(x,\ t,\ u_2)-f(x,\ t,\ u_1)|\leqslant L|u_2-u_1|$$

is satisfied, where L is a positive number. The function $\varphi(x)$ is integrable.

The difference scheme has the form

$$u_{i,\,k+1}=\sum_{r=-N}^{N}C_r(x,\ t,\ \Delta t)\,u_{i+r,\,k}+\Delta t f_{i,k}, \tag{1}$$

where N is determined by the number of points of approximation of the derivatives. As established in [1], scheme (1) will be stable if its coefficients satisfy the condition

$$\left|\sum_{r=-N}^{N}C_r(x,\ t,\ 0)\exp[ir\beta]\right|\leqslant\exp[-M\beta^2], \tag{2}$$

where $|\beta| \leq \pi$ and the constant $M > 0$.

Consider the heat transfer equation containing nonlinear terms

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-hu+\sigma(u^4-u_0^4)$$

and the conditions

$$u(x,\ 0)=\varphi(x),\quad u(-l,\ t)=u(l,\ t)=0.$$

This problem was discussed in [2] in connection with a study of the thermal inertia of thermocouples.

In this case we have the difference scheme

$$u_{i,\,k+1}=\sum_{r=-1}^{1}C_r(x,\ t,\ \Delta t)\,u_{i+r,\,k}+\sigma(u_{i,\,k}^4-u_0^4), \tag{3}$$

where we set $N = 1$, since the derivatives are approximated with respect to three points.

We will investigate the stability of the scheme obtained. We have

$$\sum_{r=-1}^{1}C_r(x,\ t,\ \Delta t)\,u_{i+r,\,k}=\frac{\Delta t}{\Delta x^2}u_{i+1,\,k}+$$

$$+\left(1-2\frac{\Delta t}{\Delta x^2}-h\,\Delta t\right)u_{i,\,k}+\frac{\Delta t}{\Delta x^2}u_{i-1,\,k}.$$

Then, setting

$$\lambda=\frac{\Delta t}{\Delta x^2},\quad C_{-1}\Big|_{\Delta t=0}=\lambda,\quad C_0\Big|_{\Delta t=0}=1-2\lambda,\quad C_1\Big|_{\Delta t=0}=\lambda,$$

we correspondingly obtain

$$\sum_{r=-1}^{1}C_r(x,\ t,\ 0)\exp[ir\beta]=\lambda(\exp[-i\beta]+$$

$$+\exp[i\beta])+1-2\lambda=1-4\lambda\sin^2\frac{\beta}{2}.$$

In this case the stability condition (2) will be

$$\left|1-4\lambda\sin^2\frac{\beta}{2}\right|\leqslant\exp[-M\beta^2],\quad |\beta|\leqslant\pi. \tag{4}$$

For its satisfaction it is necessary that

$$\left|1-4\lambda\sin^2\frac{\beta}{2}\right|\leqslant 1\quad \text{for}\quad |\beta|\leqslant\pi,$$

which holds when $\lambda \leq 1/2$.

From the series expansions

$$1-4\lambda\sin^2\frac{\beta}{2}=1-4\lambda\frac{\beta^2}{4}+\ldots,$$

$$\exp[-M\beta^2]=1-M\beta^2+\ldots$$

and inequality (4) it follows that $M \leq \lambda \leq 1/2$. Hence the difference scheme (3) is stable when $\Delta t/\Delta x^2 \leq 1/2$.

We shall now consider a problem with nonlinear boundary conditions [3, 4]. We have the linear equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},$$

and the nonlinear boundary condition

$$\frac{\partial u}{\partial x}\Big|_{x=0}=0,\quad \frac{\partial u}{\partial x}\Big|_{x=l}=(1-u^4)\Big|_{x=l}.$$

The initial condition is

$$u(x,\ 0)=\varphi(x).$$

The difference scheme of this problem is written thus:

$$u_{i,\,k+1}=(1-2\lambda)\,u_{i,\,k}+\lambda(u_{i+1,\,k}+u_{i-1,\,k}),$$

$$u_{1,k}=u_{0,k},\quad u_{n,k}=u_{n-1,\,k}+(1-u_{n,k}^4)\,\Delta x,\quad u_{i,0}=\varphi_i. \tag{5}$$

It is known that a difference scheme with linear boundary conditions is stable if $\lambda = \Delta t/\Delta x^2 \leq 1/2$ [5]. This means that the difference $\varepsilon_{i,\,k}$ between the exact solution of the difference scheme and its approximate solution at the point (x_i, t_k) remains bounded or decreases as $k \to \infty$. In scheme (5), apart from all the errors of the linear scheme, there is the error in solving an equation of fourth degree. If we show that these errors do not increase as $k \to \infty$, we will have proved the stability of scheme (5).

Of all the roots of the equation

$$u_{n,k}=u_{n-1,k}+(1-u_{n,k}^4)\,\Delta x \tag{6}$$

we are interested only in that $u^*_{n,k}$ that tends to $u_{n-1,k}$ as $\Delta x \to 0$. We will investigate the stability of the difference scheme at this value of $u^*_{n,k}$. We assume that the approximate solution of Eq. (6) has been found with an error $\delta_{n,k}$, so that

$$u^*_{n,k} = u_{n,k} + \delta_{n,k}.$$

In this case we have the linear scheme

$$\tilde{u}_{i,k+1} = (1-2\lambda)\tilde{u}_{i,k} + \lambda(\tilde{u}_{i-1,k} + \tilde{u}_{i+1,k}),$$

$$\tilde{u}_{1,k} = \tilde{u}_{0,k}, \quad \tilde{u}_{n,k} = u_{n,k} + \delta_{n,k}, \quad \tilde{u}_{i,0} = \varphi_i. \tag{7}$$

Denoting the difference

$$\tilde{u}_{i,k} - u_{i,k} = \varepsilon_{i,k},$$

we have, subtracting (5) from (7),

$$\varepsilon_{i,k+1} = (1-2\lambda)\varepsilon_{i,k} + \lambda(\varepsilon_{i-1,k} + \varepsilon_{i+1,k}),$$

$$\varepsilon_{1,k} = \varepsilon_{0,k}, \quad \varepsilon_{n,k} = \delta_{n,k}, \quad \varepsilon_{i,0} = 0. \tag{8}$$

The linear difference scheme (8) is stable for $\lambda \le 1/2$. In this case the inequality

$$\max_i |\varepsilon_{i,n}| \leqslant N \max_k |\delta_{n,k}|,$$

is satisfied [6], where $N > 0$.

Hence it follows that the error $\varepsilon_{n,k}$ does not increase as $k \to \infty$ and hence scheme (5) is also stable at $\lambda \le 1/2$.

REFERENCES

1. R. D. Richtmyer, Difference Methods for Initial Value Problems [Russian translation], IL, 1960.

2. P. V. Cherpakov, IFZh, **5**, no. 9, 1962.

3. Yu. S. Postol'nik, IFZh [Journal of Engineering Physics], **8**, no. 1, 1965.

4. V. N. Sokolov, Calculation of the Heating of Metal in Metallurgical Furnaces [in Russian], Metallurgizdat, 1956.

5. V. K. Saul'ev, Integration of Equations of Parabolic Type by the Net-Point Method [in Russian], Fizmatgiz, 1960.

6. S. K. Godunov and V. S. Ryaben'kii, Introduction to the Theory of Difference Schemes [in Russian], Fizmatgiz, 1962.

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